South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, Proceedings (2022), pp. 247-258

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

NEGATION OF CAYLEY SIGNED GRAPHS

Pranjali, Amit Kumar* and Tanuja Yadav*

Department of Mathematics,
University of Rajasthan,
JLN Marg, Jaipur - 302004, INDIA

E-mail : pranjali48@gmail.com

*Department of Mathematics and Statistics,
Banasthali Vidyapith, Banasthali-304022, INDIA

E-mail : amitsu48@gmail.com, yadav.tanuja17@gmail.com

(**Received:** *Apr. 08, 2022* **Accepted:** *Aug. 05, 2022* **Published:** *Aug. 30, 2022)*

Special Issue
Proceedings of National Conference on
"Emerging Trends in Discrete Mathematics, NCETDM - 2022"

Abstract: The aim of author's in this paper is to study negation of Cayley signed graph. Further we have characterized Cayley sets and generating sets for which the negation of Cayley signed graph is balanced and $\mathcal{C}$-consistent.

Keywords and Phrases: Cayley Graph, Generating set, Signed graph.

2020 Mathematics Subject Classification: 20D05, 05C75, 05C22.

1. Introduction

In 1878, the notion of Cayley graph was introduced by Cayley [2] to illustrate the concept of 'group' and 'generating' subsets. The formal definition is as follows: "The Cayley graph of Γ, denote by, $Cay(\Gamma, S)$ is a simple graph whose vertices are the elements of Γ, and two vertices x and y are adjacent if and only if there exists $s \in S$ such that $x = sy$". For detailed study on the Cayley graphs the reader is referred to [9, 11].

"A graph Γ equipped with a signature σ is called a signed graph, denoted by $\Sigma := (\Gamma, \sigma)$, where $\Gamma = (V, E)$ is an underlying graph and $\sigma : E \to \{+, -\}$ is the signature that labels each edge of Γ either by '+' or '$-$'. The edge which receives the '+'($-$) sign is called positive(negative) edge. A signed graph is an all-positive(all-negative) if all of its edges are positive(negative); further, it is said to be homogeneous if it is either an all-positive or an all-negative and heterogeneous otherwise. The negative degree $d^-(v)$ of a vertex v is the number of negative edges incident at v in Σ and the positive degree $d^+(v)$ is defined similarly."

One of the fundamental concept in the theory of signed graph is that of balance. Harary [5] introduced the fascinated concept of balanced signed graphs for the analysis of social networks, in which a positive edge stands for a positive relation and a negative edge is for negative relation. A signed graph is balanced if every cycle has an even numbers of negative edges. A cycle in a signed graph Σ is said to be positive if it contains an even number of negative edges. For more details on the concept of balance and consistency we referee to [3, 6]. The following is well-known criteria for balance.

Lemma 1.1. [13] *A signed graph in which every chordless cycle is positive is balanced.*

A marked signed graph is a signed graph each vertex of which is designated to be positive or negative and it is consistent if every cycle in signed graph possesses an even number of negative vertices. Consistent marked graphs were introduced by Beineke and Harary [1], and was motivated by communication networks. A marked signed graph is an ordered pair $\Sigma_\mu = (\Sigma, \mu)$, where $\Sigma = (\Gamma, \sigma)$ is a signed graph and $\mu : V(\Sigma) \to \{+, -\}$ is a function from the vertex set $V(\Sigma)$ into the set $\{+, -\}$, called marking of Σ. In particular, σ induces a unique marking μ_σ defined by

$$\mu_\sigma(v) = \prod_{e \in E_v} \sigma(e),$$

where E_v is the set of edges incident at v in Σ, is called a canonical marking of Σ. If every vertex of a given signed graph Σ is canonically marked, then a cycle Z in Σ is said to be canonically consistent ($\mathcal{C}$-consistent) if it contains an even number of negative vertices and the given signed graph Σ is said to be $\mathcal{C}$-consistent if every cycle in it is $\mathcal{C}$-consistent. Due to enormous number of applications in various fields, signed graphs are leading to vast variety of results and questions and number of papers with their applications have been published in the reputed international journals, for detail bibliography of signed graphs reader is referred to up-to-date creative survey article of Zaslavsky [12].

Throughout this article, all graphs are assumed to be simple, i.e., undirected graphs in which any two vertices are joined by at most one edge and without loops. For terminology and notations from group theory and graph theory not defined in this paper, we refer the reader to [7] and [4] respectively.

1.1. Preliminary Analysis

In this subsection, we briefly recall the definitions which are needed in the sequel.

Let Γ be an abelian group. The group of integers modulo n, denoted by $\mathbb{Z}_n$ in which the sets $Z(\mathbb{Z}_n)$ and $U(\mathbb{Z}_n)$ are defined as; $Z(\mathbb{Z}_n) = \{x : gcd(x, n) \neq 1\}$ and $U(\mathbb{Z}_n) = \{y : gcd(y, n) = 1\}$. Also, $U(\mathbb{Z}_m \times \mathbb{Z}_n)$ is defined as; $U(\mathbb{Z}_m \times \mathbb{Z}_n) = \{(x, y) : gcd(x, m) = 1 \;\&\; gcd(y, n) = 1\}$.

Definition 1.2. [2] *A nonempty subset S of Γ is called Cayley set or symmetric Cayley set if $e \notin S$ and for every $a \in S$, $a^{-1} \in S$. If Cayley set generates group Γ, then S is called generating set or symmetric generating set.*

Consequently, for a given group Γ of order n,

$$1 \leq |S| \leq n - 1. \tag{1.1}$$

However, if S generates Γ, then

$$2 \leq |S| \leq n - 1. \tag{1.2}$$

The following example illustrate the above concepts:

Example 1.3. Let $\Gamma \cong \mathbb{Z}_4$. Then possible Cayley sets are $S_1 = \{2\}$, $S_2 = \{1, 3\}$, $S_3 = \{1, 2, 3\}$ and out of them S_2 and S_3 are both generating sets.

The following concept of Cayley signed graph was initiated in [10].

Definition 1.4. *Let S be a Cayley set of a finite group Γ. The Cayley signed graph, denoted by $Cay_\Sigma(\Gamma, S) := (Cay(\Gamma, S), \sigma)$ is a signed graph whose underlying graph is $Cay(\Gamma, S)$ with vertex set Γ and generating set S, and for an edge $(x, y) \in E(Cay(\Gamma, S))$, the signature σ is defined as*

$$\sigma(x,\, y) = \begin{cases} +, & \text{if } x \in S \text{ or } y \in S; \\ -, & \text{otherwise.} \end{cases}$$

Lemma 1.5. [10] *Let Γ be a finite cyclic group and p be a prime. Then $Cay_\Sigma(\Gamma, S)$ is balanced if any one of the following condition holds:*

(i) $S \subseteq U(\Gamma)$, when $|\Gamma|$ is an even;

(ii) $S = S' \cup \{p\}$, when $|\Gamma| = 2p$, where $S' \subseteq U(\Gamma)$;

(iii) $S = \{\frac{|\Gamma|}{4}, \frac{3|\Gamma|}{4}\}$, *when* $|\Gamma|$ *is multiple of* 4;

(iv) $S = U(\Gamma)$, *when* $|\Gamma| = p^k$, $k \geq 1$;

(v) $S = \{a : \ a \neq a^{-1}, \ \forall \ a \in \Gamma\}$, *when* $|\Gamma|$ *is an even.*

2. Negation of Cayley Signed Graphs

The negation of Cayley signed graph, denoted by, $\eta(Cay_\Sigma(\Gamma, S))$ is a signed graph obtained from Cayley signed graph $Cay_\Sigma(\Gamma, S)$ by negating the sign of every edge of $Cay_\Sigma(\Gamma, S)$. To illustrate the notion we have the following example.

Example 2.1. Let $\Gamma \cong \mathbb{Z}_3$ be a finite cyclic group. Then there is only one Cayley set precisely $S_1 = \{1, 2\}$, which is also a generating set. The Cayley signed graph $Cay_\Sigma(\mathbb{Z}_3, S_1)$ and its negation $\eta(Cay_\Sigma(\mathbb{Z}_3, S_1))$ with respect to generating set S_1 are shown in Figure 1(a) and Figure 1(b), respectively.

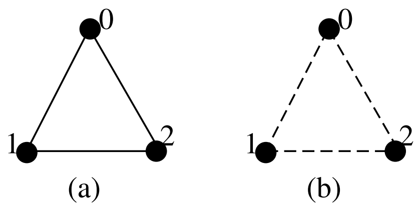

Figure 1: The Cayley signed graph and its negation

In this Paper, we study the negation of Cayley signed graphs and determine the Cayley sets and generating sets for which negation of Cayley signed graphs is balanced or $\mathcal{C}$-consistent.

Now, we will look at some examples of balanced and $\mathcal{C}$-consistent negation of Cayley signed graphs.

Example 2.2. Let $\Gamma \cong \mathbb{Z}_5$ be a finite cyclic group. Then there are three Cayley sets which are also generating set, namely, $S_1 = \{1, 4\}$, $S_2 = \{2, 3\}$ and $S_3 = \{1, 2, 3, 4\}$. The negation Cayley signed graphs $\eta(Cay_\Sigma(\mathbb{Z}_5, S_1))$, $\eta(Cay_\Sigma(\mathbb{Z}_5, S_2))$ and $\eta(Cay_\Sigma(\mathbb{Z}_5, S_3))$ are shown in Figure 2(a), Figure 2(b), and Figure 2(c), respectively.

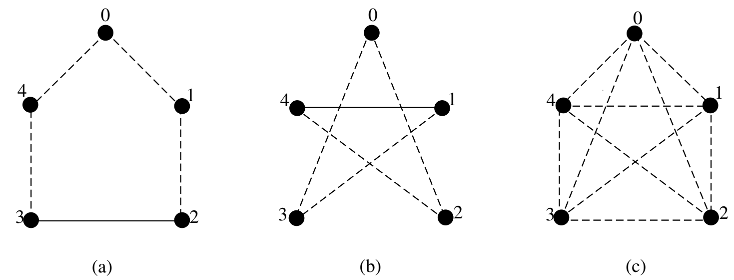

Figure 2: The negation of Cayley signed graph

Here, one can observe that the negation of Cayley signed graph with respect to generating sets S_1 and S_2 are balanced because both $\eta(Cay_\Sigma(\mathbb{Z}_5, S_1))$ and $\eta(Cay_\Sigma(\mathbb{Z}_5, S_2))$ are cycle consisting of an even number of negative edges. However, $\eta(Cay_\Sigma(\mathbb{Z}_5, S_3))$ is an all-negative signed graph and there exist an all negative triangle which indicates that $\eta(Cay_\Sigma(\mathbb{Z}_5, S_3))$ is not balanced.

Example 2.3. Let $\Gamma \cong \mathbb{Z}_4$ be a finite cyclic group. Then, there are three Cayley sets namely, $S_1 = \{2\}$, $S_2 = \{1, 3\}$ and $S_3 = \{1, 2, 3\}$. The negation of Cayley signed graphs $\eta(Cay_\Sigma(\mathbb{Z}_4, S_1))$, $\eta(Cay_\Sigma(\mathbb{Z}_4, S_2))$ and $\eta(Cay_\Sigma(\mathbb{Z}_4, S_3))$ are shown in Figure 3(a), Figure 3(b) and Figure 3(c), respectively.

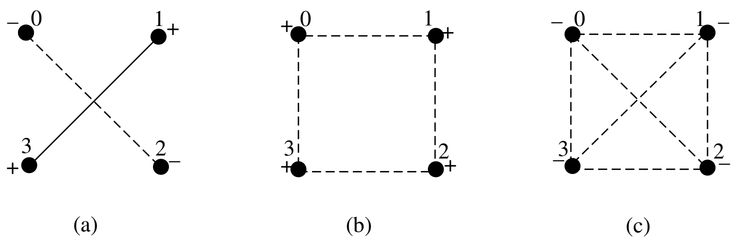

Figure 3: The canonically marked negation of Cayley signed graph

Here, one can observe that $\eta(Cay_\Sigma(\mathbb{Z}_4, S_1))$ is $\mathcal{C}$-consistent as there is no cycle in $\eta(Cay_\Sigma(\mathbb{Z}_4, S_1))$. In $\eta(Cay_\Sigma(\mathbb{Z}_4, S_2))$ all vertices receive positive sign under canonical marking. Hence, $\eta(Cay_\Sigma(\mathbb{Z}_4, S_2))$ is $\mathcal{C}$-consistent. In $\eta(Cay_\Sigma(\mathbb{Z}_4, S_3))$ all vertices receive negative sign under canonical marking. So, by taking any three vertices in $\eta(Cay_\Sigma(\mathbb{Z}_4, S_3))$ we get a cycle having an odd number of negatively marked vertices. Hence, $\eta(Cay_\Sigma(\mathbb{Z}_4, S_3))$ is not $\mathcal{C}$-consistent.

3. Balanced $\eta(Cay_\Sigma(\Gamma, S))$

In this section, we shall find some sufficient conditions on Cayley set for which $\eta(Cay_\Sigma(\Gamma, S))$ is balanced.

Theorem 3.1. *Let Γ be a finite abelian group of order n and S be a Cayley set with $|S| = 1$. Then $\eta(Cay_\Sigma(\Gamma, S))$ is balanced.*

Proof. Let Γ be a finite abelian group of order n and $|S| = 1$. Then $Cay(\Gamma, S)$ is isomorphic to $\frac{n}{2}$-copies of K_2. Clearly, due to absence of cycles, $\eta(Cay_\Sigma(\Gamma, S))$ is balanced trivially.

Lemma 3.2. *Let Γ be a finite abelian group of order n and S be a generating set with $|S| = n - 1$. Then $\eta(Cay_\Sigma(\Gamma, S))$ is not balanced.*

Proof. Let Γ be a finite abelian group and S be a generating set with $|S| = n - 1$. Then all non-zero elements of Γ belongs to S. This implies that there exist an all negative triangle in $\eta(Cay_\Sigma(\Gamma, S))$ among any three elements of Γ. Therefore, $\eta(Cay_\Sigma(\Gamma, S))$ is not balanced.

Theorem 3.3. *Let Γ be a finite abelian group of order n and S be a generating set with $|S| = n - 2$. Then $\eta(Cay_\Sigma(\Gamma, S))$ is balanced if and only if $n = 4$.*

Proof. Necessity: Let Γ be a finite abelian group of order n and S be a generating set with $|S| = n - 2$. Let us assume $\eta(Cay_\Sigma(\Gamma, S))$ is balanced and we have to prove $n = 4$. Let us assume $n \neq 4$. Then $n > 4$. As $|S| = n - 2$ and $n > 4$, then in $\eta(Cay_\Sigma(\Gamma, S))$ there exist a negative triangle among 0, $n - 1$ and $n - 2$ which shows that $\eta(Cay_\Sigma(\Gamma, S))$ is not balanced, a contradiction to the assumption. Thus by contrapositive, the value of n must be 4 only.
Sufficiency: If $n = 4$, then $|S| = 2$. In this case, $\eta(Cay_\Sigma(\Gamma, S))$ is isomorphic to an all negative 4-cycle and which shows that $\eta(Cay_\Sigma(\Gamma, S))$ is balanced.

Theorem 3.4. *Let Γ be a finite cyclic group, S be a Cayley set and p be a prime. Then $\eta(Cay_\Sigma(\Gamma, S))$ is balanced if any one of the following condition holds:*

(i) $S \subseteq U(\Gamma)$, *when* $|\Gamma|$ *is an even;*

(ii) $S = S' \cup \{p\}$, *when* $|\Gamma| = 2p$, $|\Gamma| \neq 4$, *and* $S' \subseteq U(\Gamma)$;

(iii) $S = \{\frac{|\Gamma|}{4}, \frac{3|\Gamma|}{4}\}$, *when* $|\Gamma|$ *is multiple of* 4;

(iv) $S = \{a, a^{-1}\}; a \in U(\Gamma)$, *when* $|\Gamma|$ *is odd and* $|\Gamma| > 3$.

Proof. (i) Let Γ be a finite cyclic group of even order and S be a Cayley set. If $S \subseteq U(\Gamma)$, then every negative edge has one end vertex with odd label and all edges incident to that odd labeled vertex are all negative or all positive. So,

every cycle in $Cay_\Sigma(\Gamma, S)$ has an even number of negative edges. Since, $|\Gamma|$ is an even and $S \subseteq U(\Gamma)$ this implies that length of every cycle in $Cay_\Sigma(\Gamma, S)$ is even. Therefore, each cycle in $\eta(Cay_\Sigma(\Gamma, S))$ contains an even number of negative edges. Thus $\eta(Cay_\Sigma(\Gamma, S))$ is balanced.
(ii) If $S = S' \cup \{p\}$, where $|\Gamma| = 2p$, $|\Gamma| \neq 4$, and $S' \subseteq U(\Gamma)$, then either $Cay_\Sigma(\Gamma, S)$ is an all positive signed graph or every cycle in $Cay_\Sigma(\Gamma, S)$ consists of even number of negative edges due to Lemma 1.5. Note that each cycle in $Cay_\Sigma(\Gamma, S)$ is of even length and hence in $\eta(Cay_\Sigma(\Gamma, S))$ as well. This indicates the presence of even number of negative edges in each cycle in $\eta(Cay_\Sigma(\Gamma, S))$. Thus $\eta(Cay_\Sigma(\Gamma, S))$ is balanced.
(iii) If $S = \{\frac{|\Gamma|}{4}, \frac{3|\Gamma|}{4}\}$, where $|\Gamma|$ is multiple of 4, then $Cay(\Gamma, S)$ is isomorphic to $\underbrace{C_4 \cup C_4 \cup \cdots \cup C_4}_{k-times}$ and its respective signed graph $Cay_\Sigma(\Gamma, S)$ is isomorphic to $\underbrace{C_4 \cup C_4 \cup \cdots \cup C_4}_{k-times}$ in which exactly one C_4 is an all positive and remaining C_4 are all negative. This implies that $\eta(Cay_\Sigma(\Gamma, S))$ consists of one C_4 which is an all negative and remaining C_4 are all positive. Therefore, $\eta(Cay_\Sigma(\Gamma, S))$ is balanced.
(iv) If $S = \{a, a^{-1}\}; a \in U(\Gamma)$, where $|\Gamma|$ is odd and $|\Gamma| > 3$, then $Cay_\Sigma(\Gamma, S)$ is isomorphic to cycle of odd length consisting of odd number of negative edges. This implies that $\eta(Cay_\Sigma(\Gamma, S))$ is isomorphic to a cycle of odd length consisting of even number of negative edges. Thus $\eta(Cay_\Sigma(\Gamma, S))$ is balanced.

Theorem 3.5. *Let $\Gamma \cong \mathbb{Z}_2^t$, $t \geq 1$, and $S = Z^0(\Gamma)$. Then $\eta(Cay_\Sigma(\Gamma, S))$ is balanced if and only if $t = 2$.*
Proof. Let $\Gamma \cong \mathbb{Z}_2^t$, $t \geq 1$, and $S = Z^0(\Gamma)$. This gives $|S| = n - 2$. Now in the view of Theorem 3.3, it is clear that Γ must be an abelian group of order 4. Therefore, Γ must be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, the result.

Theorem 3.6. *Let $\Gamma \cong \mathbb{Z}_2^t$, $t \geq 1$, and $S = U(\Gamma)$. Then $\eta(Cay_\Sigma(\Gamma, S))$ is balanced.*
Proof. Let $\Gamma \cong \mathbb{Z}_2^t$, $t \geq 1$, and $S = U(\Gamma)$. Then $|S| = 1$. Therefore, the result holds due to Theorem 3.1.

Theorem 3.7. *Let $\Gamma \cong \mathbb{Z}_{p^k}$, $(k \geq 1)$, where p is prime, and $S = U(\Gamma)$ be a generating set. Then $\eta(Cay_\Sigma(\Gamma, S))$ is balanced if and only if $p = 2$.*
Proof. Necessity: Let $\Gamma \cong \mathbb{Z}_{p^k}$, $(k \geq 1)$, and $S = U(\Gamma)$ be a generating set. Let $\eta(Cay_\Sigma(\Gamma, S))$ is balanced and $p \neq 2$. This implies that $|S| \geq 2$ and $Cay_\Sigma(\Gamma, S)$ is all-positive due to Lemma 1.5(iv). Thus, if $|S| \geq 2$, $\eta(Cay_\Sigma(\Gamma, S))$ is all-negative signed graph. Then there always exist an all negative triangle in $\eta(Cay_\Sigma(\Gamma, S))$ among 0, $(p^k - 1)$ and $(p^k - 2)$ and this indicates that $\eta(Cay_\Sigma(\Gamma, S))$ is not balanced, a contradiction, hence, by contrapositive result holds.

Sufficiency: Let $p = 2$. Then $\eta(Cay_\Sigma(\Gamma, S))$ is balanced due to Theorem 3.4(i).

Theorem 3.8. *Let $\Gamma \cong \mathbb{Z}_{p^k}$, $(k > 1)$, where p is prime, and $S = Z^0(\Gamma)$ be a Cayley set. Then $\eta(Cay_\Sigma(\Gamma, S))$ is balanced if and only if $p = 2$ with $k = 2$.* **Proof.**
Necessity: Let $\Gamma \cong \mathbb{Z}_{p^k}$, $(k > 1)$, and $S = Z^0(\Gamma)$ be a Cayley set. Let us assume that $\eta(Cay_\Sigma(\Gamma, S))$ is balanced and given condition is false. Then there are two cases arise :
Case 1: If $p = 2$ with $k > 2$, then there exists an all-negative triangle in $\eta(Cay_\Sigma(\Gamma, S))$ among 0, 2, and 4.
Case 2: If $p > 2$ with $k > 1$ then $p^k > 4$. Since in $\eta(Cay_\Sigma(\Gamma, S))$ the negative edges lie between the vertices whose labels are multiples of p and 0, so there exist an all-negative triangle in $\eta(Cay_\Sigma(\Gamma, S))$ among 0 and two elements of S because $p^k > 4$. Thus $\eta(Cay_\Sigma(\Gamma, S))$ is not balanced in both cases, a contradiction. Hence, by contrapositive result holds.
Sufficiency: Let $p = 2$, $k = 2$. Then $|S| = 1$ implies that $\eta(Cay_\Sigma(\Gamma, S))$ is balanced due to Theorem 3.1.

4. $\mathcal{C}$-Consistent $\eta(Cay_\Sigma(\Gamma, S))$

This section is devoted to find sufficient conditions on S, for which $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent.

Theorem 4.1. *Let Γ be a finite abelian group of order n and S be a generating set. Then $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent if any one of the following condition holds:*

(i) $|S| = 1$;

(ii) $|S| = 2$, *when* $n > 2$.

Proof. (i) Let Γ be a finite abelian group and S be a Cayley set with $|S| = 1$. Then $Cay(\Gamma, S))$ is 1-regular graph and clearly there is no cycle in $\eta(Cay_\Sigma(\Gamma, S))$. Therefore $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent.
(ii) Let Γ be a finite abelian group of order $n > 2$ and S be a Cayley set with $|S| = 2$. This implies that $Cay(\Gamma, S)$ is either a cycle graph or copies of cycle graph. In this case, $Cay_\Sigma(\Gamma, S)$ consists of all positive edges or all negative edges or a negative section. This implies that every cycle in $\eta(Cay_\Sigma(\Gamma, S))$ have an even number of negatively marked vertices under $\mathcal{C}$-marking. Hence, $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent.

Theorem 4.2. *Let Γ be a finite abelian group of order n and S be a generating set. Then $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent if any one of the following condition holds:*

(i) $|S| = n - 1$, *when n is odd;*

(ii) $|S| = n-2$, *when* n *is even.*

Proof. (i) Let Γ be a finite abelian group of odd order. Then $n-1$ is even. $Cay(\Gamma, S)$ is a regular graph of even degree and $Cay_\Sigma(\Gamma, S)$ is an all-positive signed graph. This implies that $\eta(Cay_\Sigma(\Gamma, S))$ is an all-negative signed graph in which each vertex is of even degree. Therefore, all vertices in $\eta(Cay_\Sigma(\Gamma, S))$ receive positive sign under $\mathcal{C}$-marking . Hence, $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent.
(ii) If $|S| = n-2$, then $\eta(Cay_\Sigma(\Gamma, S))$ is an all-negative signed graph. Clearly, $n-2$ is even as n is even. Thus, $\eta(Cay_\Sigma(\Gamma, S))$ is all-negative signed graph of even negative degree. Therefore, each vertex in $\eta(Cay_\Sigma(\Gamma, S))$ receives positive sign under $\mathcal{C}$-marking. Hence, $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent.

Theorem 4.3. *Let* Γ *be a finite cyclic group of even order and* S *be a Cayley set. Then* $\eta(Cay_\Sigma(\Gamma, S))$ *is* $\mathcal{C}$*-consistent if any one of the following condition holds:*

(i) S *contains all elements of* Γ *except* 0 *and* $|\Gamma|/2$*;*

(ii) $S = \{\frac{|\Gamma|}{4}, \frac{3|\Gamma|}{4}\} \cup S_1$, *where* $S_1 \subseteq U(\Gamma)$ *and* $|\Gamma|$ *is multiple of* 4.

Proof. (i) If $|\Gamma|$ is even and S contains all elements of Γ except 0 and $|\Gamma|/2$, then $Cay_\Sigma(\Gamma, S)$ is all-positive signed graph. This implies that $\eta(Cay_\Sigma(\Gamma, S))$ is all-negative regular signed graph of even degree. Thus, all vertices in $\eta(Cay_\Sigma(\Gamma, S))$ receive positive sign. Hence, $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent.
(ii) If $S = \{\frac{|\Gamma|}{4}, \frac{3|\Gamma|}{4}\} \cup S_1$, where $S_1 \subseteq U(\Gamma)$ and $|\Gamma|$ is multiple of 4. This implies that $|S|$ is even and non-zero elements outside S are odd in number because one element is self inverse. Also, the difference of any two non-zero elements outside S always belongs to S. This implies that all vertices receive positive sign in $\eta(Cay_\Sigma(\Gamma, S))$ under $\mathcal{C}$-marking. Hence, $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent.

Theorem 4.4. *Let* $\Gamma \cong \mathbb{Z}_2^t$, $t \geq 1$, *and either* $S = Z^0(\Gamma)$ *or* $S = U(\Gamma)$. *Then* $\eta(Cay_\Sigma(\Gamma, S))$ *is* $\mathcal{C}$*-consistent.*
Proof. Firstly, let $\Gamma \cong \mathbb{Z}_2^t$, $t \geq 1$, and $S = Z^0(\Gamma)$. This implies that $|S| = n-2$ and $|S|$ is even. Hence, $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent due to Theorem 4.2(ii).
Secondly, let $\Gamma \cong \mathbb{Z}_2^t$, $t \geq 1$, and $S = U(\Gamma)$. This implies that $|S| = 1$. Then the result holds due to Theorem 4.1(i).

Theorem 4.5. *Let* $\Gamma \cong \mathbb{Z}_{p^k}$, $(k \geq 1)$, *where* p *is prime, and* $S = U(\Gamma)$ *be a generating set. Then* $\eta(Cay_\Sigma(\Gamma, S))$ *is* $\mathcal{C}$*-consistent.*
Proof. Let $\Gamma \cong \mathbb{Z}_{p^k}$, $(k \geq 1)$, and $S = U(\Gamma)$ be a generating set. First if $p = 2$ with $k = 1$, then $|S| = 1$. In this way, $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent due to Theorem 4.1(i).

Next if either ($p = 2$, $k > 1$) or ($p > 2$, $k \geq 1$). Then $Cay_\Sigma(\Gamma, S)$ is all-positive by Lemma 1.5(iv). This implies that $\eta(Cay_\Sigma(\Gamma, S))$ is an all-negative signed graph. Also, $|S| = p^k - p^{k-1}$ is even as $|\Gamma| > 2$. Note that $\eta(Cay_\Sigma(\Gamma, S))$ is all-negative regular signed graph of even degree. Therefore, all vertices in $\eta(Cay_\Sigma(\Gamma, S))$ receive positive sign under $\mathcal{C}$-marking. Hence, $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent.

Theorem 4.6. *Let $\Gamma \cong \mathbb{Z}_{p^k}$, ($k > 1$), where p is prime, and $S = Z^0(\Gamma)$ be a Cayley set. Then $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent if and only if either ($p = 2$ and $k = 2$) or ($p > 2$ and $k > 1$).*

Proof. Necessity: Let $\Gamma \cong \mathbb{Z}_{p^k}$, ($k > 1$), and $S = Z^0(\Gamma)$. First let us suppose that $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent and neither ($p = 2$ and $k = 2$) nor ($p > 2$ and $k > 1$). Then $|S|$ is odd and $\eta(Cay_\Sigma(\Gamma, S))$ is odd-regular graph. Note that in $\eta(Cay_\Sigma(\Gamma, S))$ an edge is positive if and only if none of end vertices belong to S. Since the negative edges lie between multiple of p and 0, and $|S|$ is odd, so the number of negative edges incident at u, $u \in S$ are odd. Thus every vertex belonging to S receive negative sign under $\mathcal{C}$-marking. In this way, there exist a triangle with three negative vertices in which one is 0 and remaining two elements belongs to S. Thus, $\eta(Cay_\Sigma(\Gamma, S))$ is not $\mathcal{C}$-consistent, a contradiction, hence, by contrapositive result hold.

Sufficiency: Case 1: If $p = 2$, $k = 2$ and $S = Z^0(\Gamma)$. Then $|S| = 1$ implies that $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent due to Theorem 4.1(i).

Case 2: If $p > 2$ and $k > 1$, then $|S|$ is even and $\eta(Cay_\Sigma(\Gamma, S))$ is even-regular graph. Note that in $\eta(Cay_\Sigma(\Gamma, S))$ an edge is positive if and only if none of end vertices belong to S. Since, the negative edges lie between multiple of p and 0, and $|S|$ is even. This implies that the number of negative edges incident at u, $u \in S$ are even. So, each vertex belonging to S receives positive sign under $\mathcal{C}$-marking. Thus, all vertices in $\eta(Cay_\Sigma(\Gamma, S))$ receive positive sign under $\mathcal{C}$-marking. Hence, $\eta(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$-consistent.

5. Conclusion

In this paper, we have characterized the Cayley sets and generating sets for which negation of Cayley signed graph is balanced, and also, for which negation of Cayley signed graph is $\mathcal{C}$-consistent. One of the highlights of this paper is that for cyclic group $\Gamma \cong \mathbb{Z}_{2^k}$ ($k \geq 1$) and $S = U(\Gamma)$, $\eta(Cay_\Sigma(\Gamma, S))$ is balanced as well as $\mathcal{C}$-consistent. For abelian group $\Gamma \cong \mathbb{Z}_2^t$, ($t \geq 1$), and either $S = Z^0(\Gamma)$ or $S = U(\Gamma)$, then $\eta(Cay_\Sigma(\Gamma, S))$ is always $\mathcal{C}$-consistent but balanced conditionally.

Acknowledgement

Author's are thankful to the anonymas referee for his/her valuable suggestion for the refinement of the paper.

References

[1] Beineke L. W. and Harary F., Consistency in Marked Graphs, J. Math. Psychol., 18 (3) (1978), 260-269.

[2] Cayley A., Desiderata and Suggestions; The Theory of Groups: Graphical Representation, American Journal of Mathematics, 1 (2) (1878), 174-176.

[3] Cartwright D. and Harary F., Structural Balance, A Generalization of Heider's Theory, Psychological Review, 63 (1956), 277-293.

[4] Harary F., Graph Theory, Addison-Wesley Publ. Comp. Reading. MA 1969.

[5] Harary F., On the Notion of Balance of a Signed Graph, Mich. Math. J., 2 (1953), 143-146.

[6] Heider F., Attitudes and Cognitive Organizations, Psychological Review, 51 (1946), 358-374.

[7] Jacobson N., Lectures in Abstract Algebra, East-West Press P. Ltd., New Delhi, 1951.

[8] Pranjali and Kumar Amit, Algebraic Signed Graphs: A Review. In: Shrimali P. and Shah N. (eds.). Recent Advancements in Graph Theory. CRC Press, New York, (2020), 260-271.

[9] Pranjali, Kumar Amit and Yadav Tanuja, Cayley Graphs Versus Algebraic Graphs, J. Indones. Math. Soc, 27 (2) (2021), 130-136.

[10] Pranjali, Kumar Amit and Yadav Tanuja, Cayley Signed Graphs Associated with Abelian Groups, J. Appl. Math. & Informatics, 40 (3) (2022), 795-805.

[11] Witte D. and Gallian J. A., A Survey: Hamiltonian Cycles in Cayley Graphs, Discrete Mathematics North Holland, 51 (1984), 293-304.

[12] Zaslavsky T., A Mathematical Bibliography of Signed and Gain Graphs and Allied Areas, Electronic Journal of Combinatorics, Dynamic Surveys in Combinatorics, #DS8 (2012), 1-346.

[13] Zaslavsky T., Signed Analogs of Bipartite Graphs, Discrete Math., 179 (1998), 205-216.